PRELIMINARY EXAMINATION IN ANALYSIS

Part I, Real Analysis

January 6, 2014

Solve 4 of the following 5 problems.

1. Let f and g be bounded measurable functions on $\mathbb{R}^n$. Assume that g is integrable and satisfies $\int g = 0$. Define $g_k(x) = k^n g(kx)$ for $k \in \mathbb{N}$. Show that $f * g_k \to 0$ pointwise almost everywhere, as $k \to \infty$.

2. Let $0 < q < p < \infty$. Let $E \subset \mathbb{R}^n$ be measurable with measure $|E| < \infty$. Let f be a measurable function on $\mathbb{R}^n$ such that $N \overset{\text{def}}{=} \sup_{\lambda > 0} \lambda^p \big|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\big|$ is finite.

(a) Prove that $\int_E |f|^q$ is finite.

(b) Refine the argument of (a) to prove that

$$\int_E |f|^q \leq C N^{q/p} |E|^{1-q/p} \,,$$

where C is a constant that depends only on n, p, and q.

3. Is the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} x \sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$ absolutely continuous on $[0,1]$? Explain fully.

4. Consider the Hardy-Littlewood maximal function (for balls)

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f| \,, \qquad f(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1, \end{cases} \qquad x \in \mathbb{R}^n \,,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x. Prove that Mf belongs to $\in \mathrm{L}^1_{\text{weak}}(\mathbb{R}^n)$.

5. Let (X, Σ, μ) be a finite measure space and $1 \leq q < p < \infty$. Let $f_1, f_2, \ldots \in \mathrm{L}^p(X, \mu)$ with $\|f_k\|_p \leq 1$ for all k. Assuming $f_k \to f$ in measure, show that $f \in \mathrm{L}^p(X, \mu)$, and that $\|f_k - f\|_q \to 0$.