PRELIMINARY EXAMINATION IN ANALYSIS

Part II, Complex Analysis

January 6, 2014

Solve 4 of the following 5 problems.

1. Let f be an entire function and define $M(r) = \max_{|z|=r} |f(z)|$. Show that M is a continuous function on $[0, \infty)$.

2. Show that for any real number $\lambda > 1$ and any integer $n \geq 1$, the equation $z^n e^{\lambda - z} = 1$ has exactly n solutions in the unit disk $|z| \leq 1$, with exactly one being real and positive.

3. Assume that f is an entire function of finite order. Prove that if $|f(z)| \leq 1$ for all z on the boundary of the horizontal half-strip $S = \big\{z \in \mathbb{C} : \mathrm{Re}\,(z) \geq 0,\ |\mathrm{Im}\,(z)| \leq 1\big\}$, then $|f(z)| \leq 1$ for all $z \in S$.
Hint. Consider $f(z)\, e^{-\epsilon z^n}$, with n chosen appropriately and $\epsilon > 0$.

4. Suppose f is an entire function with the property that $f(z)$ is real if and only if z is real. Show that $f'(z) \neq 0$ for all real z.

5. Let $G \subset \mathbb{C}$ be open, and define $\Omega = \big\{z \in \mathbb{C} : z^4 \in G\big\}$. Assume that f is analytic on Ω and satisfies $f(iz) = f(z)$ for all $z \in \Omega$. Show that there exists an analytic function g on G, such that $f(z) = g\big(z^4\big)$ for all $z \in \Omega$.