PRELIMINARY EXAMINATION: APPLIED MATHEMATICS I

January 8, 2014, 1:00-2:30 p.m.

Work all 3 of the following 3 problems.

1. Let X and Y be Banach spaces, and let $B(X, Y)$ denote the space of bounded linear mappings of X into Y, with operator norm $|| \cdot ||_{B(X,Y)}$.

(**a**) Let I be the identity mapping of X to itself. Show that for every $A \in B(X, X)$ such that $\|A\|_{B(X,X)} < 1$, the operator $(I - A)$ is bijective. [Hint: Consider the mapping $\sum_{n=0}^{\infty} A^n$.]

(**b**) Show that $E = \{A \in B(X, X) \mid A \text{ is bijective}\}$ is open in $B(X, X)$.

(**c**) Show by counter example that $G = \{A \in B(X, Y) \mid A \text{ is injective}\}$ is generally not open in $B(X, Y)$. [Hint: Consider the map $g \in C^0[0, 1] \mapsto g\mathbf{1}_{\{x\in[\epsilon,1]\}} \in L^2(0, 1)$].

2. Let $\{f_k\}$ be a sequence bounded both in $L^2(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$. Assume that f_k converges pointwise almost everywhere to $f \in L^2(\mathbb{R}^n)$.

(**a**) Show that the entire sequence $\{f_k\}$ converges weakly to f in $L^2(\mathbb{R}^n)$. [Hint: consider first compactly supported test functions].

(**b**) If additionally $||f_k||_{L^2} \to ||f||_{L^2}$, show that the entire sequence $\{f_k\}$ strongly converges to f in $L^2(\mathbb{R}^n)$.

3. For $f \in L^2(0, \infty)$, define

$$(Tf)(x) = \frac{1}{x}\int_0^x f(s)\, ds \qquad \text{for } x \in (0, \infty).$$

(**a**) Show that T is a bounded linear operator on $L^2(0, \infty)$. [Hint: Use an integration by parts, noting that $1/x^2 = (-1/x)'$].

(**b**) Show that T is not compact by considering the sequence $f_n(x) = \sqrt{n}\mathbf{1}_{\{x\in[0,1/n]\}}$.