PRELIMINARY EXAMINATION: APPLIED MATHEMATICS II

January 8, 2014, 2:40-4:10 p.m.

Work all 3 of the following 3 problems.

1. Let X and Y be normed vector spaces, and let $[a,b]$ and (a,b) denote closed and open line segments between two given points $a, b \in X$.

(**a**) Let $f : X \to Y$ be a function which is continuous on the segment $[a,b]$ and differentiable on the segment (a,b), and let $A \in B(X,Y)$ be given. Use an appropriate Mean Value Theorem to show that

$$||f(b) - f(a) - A(b-a)||_Y \leq M||b-a||_X \quad \text{where} \quad M = \sup_{x\in(a,b)} ||Df(x) - A||_{B(X,Y)}.$$

(**b**) Let $g : X \to Y$ be a function which is continuous in X and differentiable in $X - \{a\}$. Show that, if $L := \lim_{x\to a} Dg(x)$ exists, then g is differentiable at a and $Dg(a) = L$.

(**c**) Consider $g : X \to \mathbb{R}$ where $g(x) = ||x||_X$. Show that g cannot be differentiable at $x = 0$. Moreover, if g happens to be differentiable for all $x \neq 0$, show that $\lim_{x\to 0} Dg(x)$ cannot exist.

2. Given a bounded, Lipschitz domain $\Omega \subset \mathbb{R}^d$ and data $a \in [L^\infty(\Omega)]^{d\times d}$ (symmetric, uniformly positive-definite), $b \in [L^\infty(\Omega)]^d$, $c \in L^\infty(\Omega)$ and $f \in H^{-1}(\Omega)$, consider the problem of finding $u \in H_0^1(\Omega)$ such that

$$\alpha(u,v) + \beta(u,v) = \gamma(v), \quad \forall v \in H_0^1(\Omega), \tag{1}$$

where

$$\begin{aligned} \alpha(u,v) &= (a\nabla u, \nabla v)_{L^2}, \\ \beta(u,v) &= (b \cdot \nabla u + cu, v)_{L^2}, \\ \gamma(v) &= \langle f, v\rangle_{H^{-1},H_0^1}. \end{aligned}$$

(**a**) Define carefully the linear operator A so that $(Au,v)_{H_0^1} = \alpha(u,v)$. Show that this A maps $H_0^1(\Omega)$ onto itself and is continuously invertible.
(**b**) Show that the linear operator B defined by $(Bu,v)_{H_0^1} = \beta(u,v)$ maps $H_0^1(\Omega)$ into itself and is compact. [Hint: use the fact that B is compact if its Hilbert-adjoint B^* is.]
(**c**) Show that (1) is equivalent to the operator equation $(A+B)u = F$ for an appropriate $F \in H_0^1(\Omega)$.

3. Given $I = [0, b]$, consider the problem of finding $u : I \to \mathbb{R}$ such that

$$\begin{cases} u'(s) = g(s)f(u(s)), & \text{for a.e. } s \in I, \\ u(0) = \alpha, \end{cases} \tag{2}$$

where $\alpha \in \mathbb{R}$ is a given constant, $g \in L_p(I)$, $p \geq 1$, and $f : \mathbb{R} \to \mathbb{R}$ are given functions. We suppose f is Lipschitz continuous and satisfies $f(0) = 0$.

(**a**) Consider the functional

$$F(u) = \alpha + \int_0^s g(\sigma)\, f(u(\sigma))\, d\sigma.$$

Show that F maps $C^0(I)$ into $C^0(I) \cap W^{1,p}(I)$. Moreover, show that $u \in C^0(I) \cap W^{1,p}(I)$ satisfies (2) if and only if it is a fixed point of F.

(**b**) Show that (2) has a unique solution $u \in C^0(I) \cap W^{1,p}(I)$ for any $g \in L_p(I)$ and $b > 0$.