THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

Preliminary Examination in Probability
Part II

January 10th, 2014

Problem 2.1. Let $(M_t)_{0\leq t\leq T}$ be a submartingale and let $\lambda > 0$. Show that

$$\lambda\mathbb{P}(\max_{0\leq t\leq T} M_t \geq \lambda) \leq \mathbb{E}[M_T 1_{\{\max_{0\leq t\leq T} M_t\geq\lambda\}}].$$

(1) consider a one-dimensional Brownian motion B starting at $B_0 = 0$. Let $u, v : [0,\infty) \to \mathbb{R}$ such that u is C^1, striclty increasing and $u(0) = 0$. Assume also that $v(t) \neq 0$ for each t and v has bounded variation. Show that the process

$$X_t = v(t)B_{u(t)}$$

is a semi-martingale (in its own filtration), and the martingale part is $\int_0^t v(s)dB_{u(s)}$.

(2) show that the martingale part is a Brownian motion if and only if $v^2(s)u'(s) = 1$ for each s

(3) Find u, v such that X defined above is an Ornstein-Uhlenbeck process wiht parameter β, i.e

$$dX_t = \beta X_t + d\gamma_t$$

for some Brownian motion γ.

Problem 2.2. (the range of Brownian Motion) Let B be a one-dimensional BM starting at zero. Define

$$S_t = \max_{s\leq t} B, \quad I_t = \inf_{s\leq t} B_s, \quad \theta_c = \inf\{t : S_t - I_t = c\},$$

for some $c > 0$.

(1) Show that, for each λ, the process

$$M_t = \cosh(\lambda(S_t - B_t))exp(-\frac{\lambda^2 t}{2})$$

is a martingale

(2) prove that

$$\mathbb{E}[exp(-\frac{\lambda^2\theta_c}{2})] = \frac{2}{1 + \cosh(\lambda c)}.$$