ALGEBRA PRELIMINARY EXAM: PART I

Problem 1

Let p be a prime number. Show that the p-Sylow subgroup of the symmetric group S_{np} is an abelian group of order p^n for $n < p$ and is a non-abelian group of order p^{p+1} when $n = p$.

Problem 2

Prove that every prime ideal in $\mathbb{Z}[\sqrt{-5}]$ is maximal.

Problem 3

(1) Let G be a finite group which acts transitively on a set X. For any $x \in X$ let $\mathrm{Stab}_G(x) = \{g \in G : gx = x\}$. Prove that

$$G = \cup_{x \in X} \mathrm{Stab}_G(x)$$

if and only if $X = \{x\}$ is a singleton and $gx = x$ for all $g \in G$, i.e. the action is trivial.

(2) Give an example of (an infinite) group G where the above fails.

Hint: Every matrix is conjugate to an upper triangular matrix over $\mathbb{C}$.

Problem 4

Let k be a field, n a positive integer, and T the linear transformation on k^n defined by

$$T(x_1, x_2, \ldots, x_n) = (x_n, x_1, x_2, \ldots, x_{n-1}).$$

We view k^n as a $k[x]$-module with x acting as T.

(1) Show that the $k[x]$-module k^n is isomorphic to $k[x]/(x^n - 1)$.

(2) Let V be a linear subspace of k^n satisfying $T(V) \subset V$. Prove that there exists a monic polynomial $g(x) \in k[x]$ dividing $x^n - 1$ such that V corresponds to

$$\{g(x)a(x) \mid a(x) \in k[x], \deg a(x) < n - \deg g(x)\}$$

under the above isomorphism.

(3) Take $k = \mathbb{R}$, the real numbers, and $n = 3$. Describe explicitly all subspaces V of $\mathbb{R}^3$ satisfying $T(V) \subset V$.

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