ALGEBRA PRELIMINARY EXAM: PART II

Problem 1

Let K be a field and $f(x), g(x) \in K[x]$ be irreducible quadratic polynomials. Show that $K[x,y]/(f(x), g(y))$ is a field if and only if $K[x]/(f(x))$ and $K[x]/(g(x))$ are non-isomorphic.

Problem 2

Let $\mathbb{Q}$ be the field of rational numbers and ζ a primitive 9-th root of unity in an algebraic closure of $\mathbb{Q}$.

(1) Show that $\mathbb{Q}(\zeta)$ has a subfield K with $K/\mathbb{Q}$ Galois of degree 3.
(2) Find a polynomial $f(x)$ of degree 3 and integer coefficients whose splitting field is K.
(3) Let p be a prime and $g(x) \in \mathbb{F}_p[x]$ of degree 3 and non-zero discriminant such that $g(x) \equiv f(x) \pmod{p}$. Show that if $p \equiv -1 \pmod{9}$ then $g(x)$ has all its roots in $\mathbb{F}_p$.

Problem 3

Let $f(x), g(x) \in \mathbb{F}_p[x]$, where $\mathbb{F}_p$ is the finite field with p elements. Suppose that

$$\max\{\deg f, \deg g\} < p.$$

Prove that $\mathbb{F}_p(x)$ is a separable extension of $\mathbb{F}_p(\frac{f(x)}{g(x)})$.

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