Numerical Analysis Prelim, Part II (spring material) August 18, 2014

1. Consider the following ODE initial value problem and the related class of numerical algorithms,

$$
\begin{aligned}
&y'(t) = f(y(t)), \quad t > 0,\, y(0) = y_0, \\
&y_{n+1} = y_n + h((1-\alpha)f(y_n) + \alpha f(y_{n+1})) \\
&\quad 0 \le \alpha \le 1,\, t_n = nh,\, n = 0,1,..
\end{aligned}
$$

(a) Determine the order of the local truncation error as a function of α and check if the Dahlquist stability condition is satisfied.
(b) For which α-values is the algorithm A-stable.
(c) Show that for α = ½ and f = iωy with real ω, the algorithm is an exact approximation of an ODE with slightly different real ω.

2. Given the parabolic equation below,

$$
\begin{aligned}
&u_t = (a(x)u_x)_x - b(x)u, \quad 0 < x < 1,\, 0 < a \le a(x) \le A,\, 0 < b \le b(x) \le B \\
&u(x,0) = u_0(x),\; u(0,t) = 0,\, u_x(1,t) = 0.
\end{aligned}
$$

(α) Develop the general form of a finite element approximation (FEM) and a discontinuous Galerkin approximation (DG) of this parabolic problem. Use trapezoidal rule (Crank-Nicolson) for time discretization.
(b) Show that the bilinear form in the finite element approximation of the related stationary problem $-(a(x)u_x)_x + b(x)u = f(x),\; u(0) = 0,\, u_x(1) = 0$ is coersive and continuous.
(c) Show that if explicit time discretization is used then the overall DG scheme will be explicit but not the FEM approximation.

3. (a) Develop a Lax-Friedrichs and also an upwind finite difference approximation of the hyperbolic system below. (Simplifying the problem by replacing the matrix A by the scalar a > 0 will give partial credit.)

$$
\begin{aligned}
&u_t = Au_x + f(x) \\
&A = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}
\end{aligned}
$$

(b) Determine the order of accuracy of the upwind approximation based on the local truncation error.
(b) Use von Neumann analysis to show that with periodic boundary conditions the Lax-Friedrichs scheme is L^2 stable.