The University of Texas at Austin
Department of Mathematics

The Preliminary Examination in Probability
Part I

Friday, Aug 22, 2014

(*Note:* Remember that a random variable ξ is called a **coin toss** if $\mathbb{P}[\xi = 1] = \mathbb{P}[\xi = -1] = \frac{1}{2}$.)

Problem 1.

(1) (*Chernoff-Hoeffding bounds for coin tosses*). Let $\{\xi_n\}_{n\in\mathbb{N}}$ be an iid sequence of coin tosses. Show that $\mathbb{E}[e^{t\xi_1}] \le \exp(\frac{1}{2}t^2)$ for all $t > 0$ and use it to prove that, for all $a > 0$ and $n \in \mathbb{N}$, we have

$$\mathbb{P}\Big[\sum_{i=1}^n \xi_i \ge a\Big] \le e^{-a^2/(2n)} \text{ and } \mathbb{P}\Big[\Big|\sum_{i=1}^n \xi_i\Big| \ge a\Big] \le 2e^{-a^2/(2n)}$$

(2) (*Random matrices*). Given $n, m \in \mathbb{N}$, let $\boldsymbol{A}$ be an $n \times m$ matrix with entries in the set $\{0,1\}$, and let $\boldsymbol{X}$ be a random vector whose components $\boldsymbol{X} = (X_1, \dots, X_m)$ are independent coin tosses. Consider the random variable $||\boldsymbol{Y}||_\infty = \max_{i=1,\dots,n} |Y_i|$, where the vector $\boldsymbol{Y} = (Y_1, \dots, Y_n)$ is given by $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X}$. Show that

$$\mathbb{P}\Big[||\boldsymbol{Y}||_\infty \ge \sqrt{4m\log n}\Big] \le \tfrac{2}{n}.$$

(*Hint:* Establish an inequality of the same type for each $i = 1, \dots, n$: consider separately the cases where the number of 1s in the i-th row of $\boldsymbol{A}$ is above or below $\sqrt{4m\log n}$.)

Problem 2 (Randomness out of thin air)**.** Let $\{(X_n, Y_n)\}_{n\in\mathbb{N}}$ be a sequence of random vectors (possibly defined on different probability spaces) converging weakly to the random vector (X, Y). Show, by means of an example, that it is possible that $Y_n \in \sigma(X_n)$ for all $n \in \mathbb{N}$, but $Y \notin \sigma(X)$. (*Hint:* Take X_n uniform on $[0,1)$ and let Y_n be the n-th digit in the binary expansion of X_n)

Problem 3 (Conditional expectation is not a projection in $\mathbb{L}^1$)**.** Let X and Y be two square-integrable random variables (on the same probability space).

(1) If $f(x) = x^2$, show that $\mathbb{E}\Big[f\Big(X - \mathbb{E}[X|\sigma(Y)]\Big)\Big] \le \mathbb{E}\Big[f(X - h(Y))\Big]$ for any Borel $h : \mathbb{R} \to \mathbb{R}$ with $h(Y)$ square integrable.

(2) Show that (1) above no longer holds true if we take $f(x) = |x|$. (*Hint:* A 3-element probability space will suffice.)