Preliminary Examination in Differential Topology: August 2014

Instructions: Do all three questions.

Time Limit: 90 minutes.

1. Let X and Y be C^∞ manifolds, and $\phi, \psi \colon X \to Y$ smooth maps.

a. If the derivative $D_x\phi$ is injective at $x \in X$, show, starting from the inverse function theorem, that there is a neighborhood U of x such that
(i) $M := \phi(U)$ is a submanifold of Y; and
(ii) $T_{\phi(x)}M$ is the image of $D_x\phi$ in $T_{\phi(x)}Y$.

b. Prove that the graph

$$\Gamma(\psi) := \{(x, y) \in X \times Y : y = \psi(x)\}$$

is a submanifold of $X \times Y$, and identify its tangent space at $(x, \psi(x))$.

2. An n-dimensional manifold M is called *parallelizable* if it supports an n-tuple $(v_1, \ldots, v_n)$ of vector fields such that for every $x \in M$, the tangent vectors $(v_1(x), \ldots, v_n(x))$ form a basis for T_xM. Which of the following manifolds are parallelizable? Justify your answers.

a. The n-torus $\mathbb{R}^n/\mathbb{Z}^n$ (where $\mathbb{Z}^n$ is the subgroup of the additive group $\mathbb{R}^n$ consisting of points whose coordinates are all integers);

b. the sphere S^2;

c. the real projective plane $\mathbb{R}P^2$.

d. Show that $S^1 \times S^{n-1}$ is parallelizable for all n. [*Hint*: consider the tangent spaces to $\mathbb{R}^n$ at points of S^{n-1}.]

3. Define

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6$$

as a 2-form on $\mathbb{R}^6$. Show that no diffeomorphism $\phi \colon \mathbb{R}^6 \to \mathbb{R}^6$ which satisfies $\phi^*\omega = \omega$ can map the unit sphere S^5 to a sphere of radius $r \neq 1$. [*Hint:* consider $\omega \wedge \omega \wedge \omega$.]