ALGEBRA PRELIMINARY EXAM: PART II

Each problem is worth 10 points and the passing score is 20 points.

Problem 1

Let $\alpha = 1 + \sqrt[3]{3} + \sqrt[3]{9}$

a) Determine the degree of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.
b) Prove that $\sqrt{3} \notin \mathbb{Q}(\alpha)$.
c) Determine the minimal polynomial of α over $\mathbb{Q}$.
d) Determine a primitive generator of the Galois closure of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.

Problem 2

Consider $f(x) = x^4 - 5x^2 + 7 \in \mathbb{Q}[x]$. Let $E/\mathbb{Q}$ be the splitting field of $f(x)$ and $\beta \in E$ be a root of $f(x)$.

a) Prove that $f(x)$ is irreducible in $\mathbb{Q}(x)$.
b) Prove that $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to the dihedral group of order 8.
c) Prove that $\beta \notin \mathbb{Q}(\zeta_n)$ for any $n \in \mathbb{N}$. Here ζ_n denotes a primitive n-th root of unity.
d) Determine the number of subfields of E which are Galois over $\mathbb{Q}$. (Justify your reasoning.)

Problem 3

Let p be a prime, $\mathbb{F}_p$ be the field with p elements.

a) Determine the number of irreducible polynomials of degree p in $\mathbb{F}_p[x]$. (Justify your reasoning.)
b) Prove that $f_a = x^p - x + a \in \mathbb{F}_p[x]$ is irreducible for every $a \in \mathbb{F}_p \setminus \{0\}$.

Date: August 20, 2025.