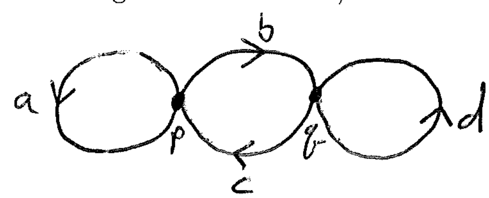
ALGEBRAIC TOPOLOGY PRELIM EXAM

Do all three problems. You have three hours. If you have any questions, please ask.

(1) Let X be the figure shown below, with two vertices and four edges.



   (a) Find explicit generators and relations for $\pi_1(X, p)$.
   (b) Find two index-3 subgroups of $\pi_1(X, p)$, one normal and one not normal.
   (c) Sketch the corresponding covering spaces of X (one regular and one not regular).

(2) Let X be a CW complex consisting of a vertex p, two edges a and b, and two 2-cells A and B, and with the following gluing maps:
   - Both endpoints of a are identified with p, as are both endpoints of b.
   - The boundary of A wraps around a, then b, then a. That is, $\partial A = aba$.
   - $\partial B = ab^{-1}a$.

   (a) Compute the fundamental group of X relative to the base point p.
   (b) Compute the homology groups of X.
   (c) Let Y be obtained from X by adding another 2-cell C with $\partial C = b$. Compute the fundamental group and homology groups of Y.

(3) Recall that a pair (X, A) has the *homotopy extension property* if and only if $\big(X \times \{0\}\big) \cup \big(A \times [0, 1]\big)$ is a retract of $X \times [0, 1]$. Also recall that the Hawaiian earring H is the union in $\mathbb{R}^2$ of the circles of radius $1/n$ centered at $(1/n, 0)$, where n ranges over the natural numbers $\mathbb{N}$. Let X be the wedge of countably many circles: $X = S^1 \times \mathbb{N}/\sim$, where a specific point on each circle (say, $\theta = 0$) is identified with the corresponding point on every other circle.
   (a) Show that (X, p) has the homotopy extension property.
   (b) Show that $(H, 0)$ does not have the homotopy extension property, where 0 is the origin in $\mathbb{R}^2$.

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