PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part I

Wednesday, August 20, 2025, 12:00 am-2:00 pm

Work all 3 of the following 3 problems.

1. The Riesz representation theorem states that if $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space and $L \in H^*$, then there is a unique $y \in H$ such that

$$Lx = \langle x, y\rangle \quad \forall x \in H.$$

Moreover, $\|L\|_{H^*} = \|y\|_H$.

(**a**) Prove the uniqueness of y.

(**b**) Prove the existence of y. [Hint: consider $N = \ker(L)$ and find $y \in N^\perp$.]

(**c**) Prove that $\|L\|_{H^*} = \|y\|_H$.

2. Let X be a Banach space.

(**a**) Define what it means for X to be separable.

(**b**) For X, define its dual space X^*.

(**c**) Prove that if X^* is separable, then X is separable. [Hint: use one of the many corollaries of the Hahn-Banach theorem.]

3. Let H be a separable Hilbert space, $\{e_n\}_{n=1}^\infty$ be a maximal orthonormal set or Hilbert basis, and $\{\lambda_n\}_{n=1}^\infty$ be a bounded sequence of real numbers. For any $x \in H$, let the operator $A : H \to H$ be given by

$$Ax = \sum_{n=1}^\infty \lambda_n \langle x, e_n\rangle e_n.$$

(**a**) Show that A is a well-defined, bounded linear operator that is also self-adjoint.

(**b**) Find the point spectrum of A, and for each distinct eigenvalue, find its eigenspace.

(**c**) Find a reasonable condition on $\{\lambda_n\}_{n=1}^\infty$ so that A is surjective.

(**d**) Show that if $\lambda_n \to 0$, then A is compact.