

**PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part II**

Friday, August 22, 2025, 12:00pm-2:00pm

Work all 3 of the following 3 problems.

1. For any Schwartz function $f \in \mathcal{S}(\mathbb{R}^3)$, we define

$$Lf(x) = f * H(x) = \int_{\mathbb{R}^3} f(y)H(x-y) dy, \text{ where } H(x) = \frac{1}{|x|^2}.$$

- (a) Show that the Fourier transform of H , denoted $\hat{H}$, is well defined as a tempered distribution. Moreover, show that there exist a function G such that for almost every $\xi \in \mathbb{R}^3$,

$$\hat{H}(\xi) = G(|\xi|).$$

- (b) Show that for any $\lambda > 0$,

$$G(\lambda) = G(1)/\lambda.$$

(You can use the formula of the Fourier transform to compute $\hat{H}(\lambda e)$ for $|e| = 1$.)

- (c) Show that there exists a constant $C > 0$ such that for any $f \in \mathcal{S}(\mathbb{R}^3)$:

$$\|\nabla(Lf)\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{L^2(\mathbb{R}^3)}.$$

2. Let

$$F(u) = \int_0^1 [(u'(x))^2 - 1]^2 dx.$$

- (a) Find all extremals in $C^1([0, 1])$ such that $u(0) = 0$ and $u(1) = 1$.

- (b) Decide if any extremal from (a) is a minimum of F .

3. Let X and Y be Banach spaces.

- (a) Let F and G take X to Y be C^1 on X , and let $H(x, \varepsilon) = F(x) + \varepsilon G(x)$ for $\varepsilon \in \mathbb{R}$. If $H(x_0, 0) = 0$ and $DF(x_0)$ is invertible, show that there exists $x \in X$ such that $H(x, \varepsilon) = 0$ for ε sufficiently close to 0.

- (b) For small ε , prove that there is a solution $w \in H^2(0, \pi)$ to

$$w'' = w + \varepsilon w^2, \quad w(0) = w(\pi) = 0.$$