DIFFERENTIAL TOPOLOGY EXAM

AUGUST 19, 2025

You should attempt all three problems. The exam length is two hours. If a question is unclear, please ask. Although you should strive for neat and detailed solutions, we are mindful that the exam is timed.

Problem 1. Consider the subset S of $\mathbb{CP}^2$ defined by

$$S = \{[X : Y : Z] \in \mathbb{CP}^2 \colon X^2 + Y^2 + Z^2 = 0\}.$$

(a) Prove that S is a smooth submanifold of $\mathbb{CP}^2$. You may freely identify real and complex derivatives. (For example, the derivative of a function from $\mathbb{C}$ to $\mathbb{C}$ may be interpreted either as a complex number, or its corresponding 2×2 real matrix.)
(b) Prove that S intersects the copy of $\mathbb{CP}^1$ in $\mathbb{CP}^2$ given by $\{[X : Y : Z] \in \mathbb{CP}^2 \colon Z = 0\}$ transversely in two points. Show that these two intersection points have the same sign.
(c) Use this to show S is not homotopic to a point.

Problem 2. In this problem, use the definition of the Lefschetz number as given in differential topology (as the sum of local Lefschetz numbers). Define the Euler characteristic $\chi(X)$ of a (compact, oriented) manifold X as the Lefschetz number of the identity map.

Let X be a (compact, oriented) manifold. Let $f \colon X \times X \to X \times X$ be the smooth map

$$f(x, y) = (y, x).$$

(a) Give the definition (or an equivalent condition) of what it means for a fixed point to be Lefschetz and how to determine the sign ($+1$ or -1) of such a fixed point.
(b) Is f Lefschetz? Explain.
(c) Calculate the Lefschetz number of f in terms of $\chi(X)$.

Problem 3. Consider the 1-form on $\mathbb{R}^2$ given by

$$\alpha = xdy - ydx$$

and let S^1 be the unit circle in $\mathbb{R}^2$.

(a) Calculate the integral of α over S^1 in two ways: first directly (by parameterizing S^1) and second by using Stokes' theorem.
(b) Does there exist a 1-form β, defined and *closed* on $\mathbb{R}^2$, such that $\alpha = \beta$ on S^1? Give an example (written in Cartesian coordinates) or prove that none exists.
(c) Does there exist a 1-form β, defined and *closed* on $\mathbb{R}^2 - \{(0, 0)\}$, such that $\alpha = \beta$ on S^1? Give an example (written in Cartesian coordinates) or prove that none exists.