Probability Theory I - Prelim Exam

Aug 19, 2025, 12:00pm - 2:00pm

Problem 1. Let X be a random variable and let $\{Y_n\}_{n\in\mathbb{N}}$ be sequence of normally distributed random variable with mean 0 and variance $\frac{1}{n}$, independent of X.

1. Prove that $X + Y_n$ is an absolutely continuous random variable for each $n \in \mathbb{N}$.
2. Prove that $X + Y_n \to X$ in distribution as $n \to \infty$.
3. Does the convergence above hold a.s.?

Problem 2. Let $\{X_n\}_{n\in\mathbb{N}}$ be an iid sequence of strictly positive random variables such that $X_1, X_1^{-1} \in \mathbb{L}^\infty$, and let $Y_n = \prod_{k=1}^n X_k$. Show that $\lim_n \frac{Y_n}{\mathbb{E}[Y_n]} = 0$ a.s. if and only if X_1 is not constant a.s.

Problem 3. Let X, Y_1, Y_2 be three random variables defined on the same probability space. Show that (X, Y_1) and (X, Y_2) have the same distribution if and only Y_1 and Y_2 have the same conditional distribution given $\sigma(X)$.