Preliminary Examination
Analysis – Part 1

August 18, 2025

Solve *only* four of the following six exercises.

Exercise 1

Prove the first Littlewood Principle which states the following: suppose that $E \subset \mathbb{R}$ is Lebesgue measurable and has finite measure, then for every $\varepsilon > 0$ there exists a finite union of open intervals O such that $\lambda(O \Delta E) < \varepsilon$ (where λ is the Lebesgue measure on $\mathbb{R}$). Recall that $O \Delta E = (O \setminus E) \cup (E \setminus O)$.

Hint: Use properties of the Lebesgue measure and the fact that an open set of $\mathbb{R}$ *is at most countable union of disjoint open intervals.*

Exercise 2

Show the Lebesgue's Dominated Convergence Theorem in $L^1(X, \mathcal{M}, \mu)$ which states the following: let $\{f_i\}$ be a sequence of measurable functions on the measure space $(X, \mathcal{M}, \mu)$ which converges μ-a.e. to a measurable function f. Assume there exists $g \in L^1(X, \mathcal{M}, \mu)$ such that $|f_i| \leq g$ μ-a.e. for any i. Then f_i, $f \in L^1(X, \mathcal{M}, \mu)$ and

$$\lim_{i \to \infty} \|f_i - f\|_1 = 0.$$

Exercise 3

Prove the Chebyshev's inequality: if $f \in L^p(X, \mathcal{M}, \mu)$, $1 \leq p < \infty$, and $t > 0$, then

$$\mu(\{|f| > t\}) \leq t^{-p} \|f\|_p^p.$$

Exercise 4

Let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuously differentiable function and let λ_f be the corresponding Lebesgue-Stieltjes measure. Prove

i) $\lambda_f << \lambda$ (where λ is the Lebesgue measure on $\mathbb{R}$).

ii) $\frac{d\lambda_f}{d\lambda} = f'$.

Exercise 5

Let $f \in L^p(X, \mathcal{M}, \mu)$, $1 \leq p < \infty$. Prove that

$$\|f\|_p = \sup \left\{ \int_X fg \, d\mu \, : \, \|g\|_{p'} = 1 \right\},$$

where $p' \in (1, \infty]$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Exercise 6

Let $f \in L^1_{loc}(\mathbb{R}^n)$. Prove for each fixed $r > 0$ that

$$\fint_{B(x,r)} |f| \, d\lambda$$

is a continuous function of x (where λ is the Lebesgue measure on $\mathbb{R}^n$).