ALGEBRA PRELIMINARY EXAM: PART II

The passing score is 20 points.

Problem 1 [8 points]

Let K/F be a Galois extension and $g \in \mathrm{Gal}(K/F)$. Compute the characteristic polynomial of g, where g is considered as an F-linear map from K to K.
(Hint: start by considering the case where $\mathrm{Gal}(K/F)$ is cyclic.)

Problem 2 [10 points]

Let p be a prime and $\mathbb{F}_p$ denote the field with p elements.

a) Prove that $(x^2 - 2)(x^2 - 5)(x^2 - 10)$ has a root over $\mathbb{F}_p$ for every prime p but it does not have a root over $\mathbb{Z}$.
b) Prove that $x^4 - 10x^2 + 1$ is irreducible in $\mathbb{Z}[x]$ but not in $\mathbb{F}_p[x]$ for any prime p. (Hint: think about the structure of finite field extensions.)

Problem 3 [10 points]

Let $n \in \mathbb{N}$ and $F/\mathbb{Q}$ be the extension generated by the n-th roots of unity.

a) Prove that the primitive n-th roots of unity form a basis of F as a vector space over $\mathbb{Q}$ if and only if n is squarefree (i.e. not divisible by the square of any integer).
b) Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of F.

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