Preliminary Examination
Analysis – Part 1

January 12, 2026

Solve *only* four of the following six exercises.

Exercise 1

Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Prove that almost uniform convergence implies convergence almost everywhere. The converse implication is also true provided the measure space $(X, \mathcal{M}, \mu)$ is finite, that is, $\mu(X) < \infty$ (Egorov's Theorem). Show, through an example, that Egorov's Theorem does not hold in general if the measure space is not finite.

Exercise 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Prove that for every $\varepsilon > 0$ there is a closed set F such that $\lambda_1(F^c) < \varepsilon$ and f is continuous on F in the relative topology. Here λ_1 denotes the Lebesgue measure on $\mathbb{R}$. Note that $\lambda_1(\mathbb{R}) = \infty$.

Hint: Apply Lusin's Theorem in any interval $(i, i+1)$, $i \in \mathbb{Z}$.

Exercise 3

Prove the Chebyshev's inequality: if $f \in L^p(X, \mathcal{M}, \mu)$, $1 \leq p < \infty$, and $t > 0$, then

$$\mu(\{|f| > t\}) \leq t^{-p} \|f\|_p^p.$$

Exercise 4

State the Radon-Nikodyn Theorem. Show, through an example, that the conclusion of the theorem may fail if the measures are not assumed to be σ-finite.

Hint: use the Lebesgue measure and the counting measure on $[0, 1]$.

Exercise 5

Let $(X, \mathcal{M}, \mu)$ be a complete σ-finite measure space, and let f be a non-negative and measurable function. Prove that

$$\int_X f \, d\mu = \int_{[0,\infty)} \mu\{x \in X \,:\, f(x) > t\} \, d\lambda_1(t),$$

where λ_1 is the Lebesgue measure on $\mathbb{R}$.

Hint: Use Tonelli's Theorem.

Exercise 6

Let f be a non-decreasing function on the interval $[a, b]$. Then f' exists λ_1-a.e. and is Lebesgue measurable on $[a, b]$, where λ_1 denotes the Lebesgue measure on $\mathbb{R}$. Show that

$$\int_a^b f' \, d\lambda_1 \leq f(b) - f(a).$$

Provide an example for which the inequality is strict.