# DIFFERENTIAL TOPOLOGY EXAM

JANUARY 15, 2026

You should attempt all three problems. The exam length is two hours. If a question is unclear, please ask. Although you should strive for neat and detailed solutions, we are mindful that the exam is timed.

**Problem 1.** Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ and $g\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$f(x,y,z) = (x + y^2 - z^2, x - y^2 + z^2) \quad \text{and} \quad g(s,t) = (s^2, s+t, s-t).$$

(a) Let $X = f^{-1}(5,-3)$. Prove that $X$ is a submanifold of $\mathbb{R}^3$.
(b) Let $Y = \operatorname{im} g$. Prove that $g$ is an embedding and thus that $Y$ is a submanifold of $\mathbb{R}^3$. Explain why $g$ is a proper map, and why this is relevant.
(c) Show that $X$ and $Y$ intersect transversely at two points.

**Problem 2.** In this question, you should use the definition of the Lefschetz number as the sum of local Lefschetz numbers.

(a) Let $\mathrm{id}\colon \mathrm{SO}(n) \to \mathrm{SO}(n)$ be the identity map. Calculate the Lefschetz number of id. *Hint:* the fact that $\mathrm{SO}(n)$ is a group may be helpful.
(b) Let $\mathrm{sq}\colon \mathrm{SO}(n) \to \mathrm{SO}(n)$ be the squaring map sending $A \mapsto A^2$. Calculate the Lefschetz number of sq.
(c) Give an example of a manifold $X$, together with two Lefschetz maps $f$ and $g$ from $X$ to itself, such that $f$ and $g$ are isotopic but $f$ and $g$ do *not* have the same Lefschetz number. *Hint:* under what conditions on $X$ is the Lefschetz number an isotopy invariant?

**Problem 3.** Throughout this question, you should not use algebraic topology, nor should you use pre-existing computations of de Rham cohomology.

(a) Using only the fundamental theorem of calculus and the definition of a closed form, prove that every closed 1-form on $\mathbb{R}^2$ is exact, as follows. Let $\alpha$ be a closed 1-form on $\mathbb{R}^2$. Let $h(x,y)$ be the integral of $\alpha$ along the horizontal path from the origin to $(x,0)$, plus the integral of $\alpha$ along the vertical path from $(x,0)$ to $(x,y)$. Show $\alpha = dh$.

Now consider the torus $T^2 = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$. Let $\gamma_1$ be any curve of the form $S^1 \times \{*\}$ and $\gamma_2$ be any curve of the form $\{*\} \times S^1$.

(b) Let $\alpha$ be a closed 1-form on $T^2$. Show that $\alpha$ is exact if and only if

$$\int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha = 0.$$

*Hint:* one direction is easy. For the other direction, what happens when we try to replicate our argument from (a)?
(c) The de Rham cohomology $H^1_{\mathrm{dR}}(T^2)$ is defined to be the space of closed 1-forms on $T^2$, modulo the subspace of exact 1-forms on $T^2$. Verify that the map $H^1_{\mathrm{dR}}(T^2) \to \mathbb{R}^2$ sending

$$[\alpha] \mapsto \left( \int_{\gamma_1} \alpha, \int_{\gamma_2} \alpha \right)$$

is a well-defined linear isomorphism.