PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part II

Friday, January 16, 2026, 6:30pm-8:30pm

Work all 3 of the following 3 problems.

1. Consider a connected, bounded domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary.

(**a**) Let u_n be a bounded sequence of $H^1(\Omega)$. Assume that ∇u_n converges to 0 in $L^2(\Omega)$. Show that u_n converges strongly in $L^2(\Omega)$ to a constant a.

(**b**) In addition, assume that $\gamma_0(u_n)$, the trace of of u_n on $\partial\Omega$, converges to 0 in $L^2(\partial\Omega)$. Show that u_n converges to 0 in $L^2(\Omega)$.

(**c**) Show that there exists $C > 0$, depending only on Ω, such that for any $u \in H^1(\Omega)$:

$$\|u\|_{L^2(\Omega)}^2 \leq C(\|\nabla u\|_{L^2(\Omega)}^2 + \|\gamma_0(u)\|_{L^2(\partial\Omega)}^2). \tag{1}$$

2. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary, and $f \in L^2(\Omega)$. Consider the boundary value problem:

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} + u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

(**a**) Formulate a variational principle for this problem.

(**b**) Show that this problem has a unique weak solution.

3. Consider the differential equation

$$\begin{aligned} &u'(t) + u(t) = \sin(u(t)) \qquad \text{for } t > 0, \\ &u(0) = u^0 \in \mathbb{R}. \end{aligned}$$

Show that there exists exactly one solution u for any initial value $u^0 \in \mathbb{R}$.