PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part I

Wednesday, January 14, 2026, 6:30 pm-8:30 pm

Work all 3 of the following 3 problems.

1. Let X be a normed linear space and $Y \subset X$ a closed subspace such that $Y \neq X$. Let θ be given with $0 < \theta < 1$.

(**a**) Prove that there is some $z \in X$ such that $\|z\| = 1$ and $\operatorname{dist}(z, Y) \geq \theta$.

(**b**) Prove that if the dimension of X is infinite and M is a closed bounded set with nonempty interior, then M is not compact.

2. Let the normed linear space X be *uniformly convex.* This means that for any ϵ, $0 < \epsilon \leq 2$, there exists $\delta > 0$ such that for all $x, y \in X$, $\|x\| = \|y\| = 1$, $\|x - y\| \geq \epsilon$, it holds that

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

Suppose that $x_n \in X$ and $x_n \rightharpoonup x$ as $n \to \infty$.

(**a**) If $\|x_n\| = \|x\| = 1$, show that $x_n \to x$. [Hint: First show that $\|x_n + x\| \to 2$.]

(**b**) If $\|x_n\| > 0$, $\|x\| > 0$, and $\|x_n\| \to \|x\|$, show that $x_n \to x$.

(**c**) If merely $\|x_n\| \to \|x\|$, show that $x_n \to x$.

3. Define the linear operator $T : L^2([0,1]) \to L^2([0,1])$ by

$$Tf(x) = \int_0^x \int_y^1 f(z)\, dz\, dy.$$

(**a**) Show that T is self-adjoint.

(**b**) Show that T is compact.

(**c**) Find an orthogonal basis for $L^2([0,1])$ based on the eigenvalues of this operator. [Hint: differentiate twice and consider carefully the boundary conditions that must be satisfied.]