Probability Theory I - Prelim Exam

Jan 12, 2026, 6:30pm - 8:30pm

Problem 1. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of square integrable random variables. Suppose that $\mathbb{E}[X_i] = 0$ for all i and that there exists a function $r : \mathbb{N}_0 \to [0,\infty)$ such that $\lim_{n\to\infty} r(n) = 0$ and

$$\mathbb{E}\big[X_i X_j\big] \leq r(j-i) \quad \text{for} \quad j \geq i.$$

Show that $\lim_n \frac{1}{n}\sum_{i=1}^{n} X_i = 0$ in $\mathcal{L}^2$ and in probability.

Problem 2. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of symmetric probability measures on $\mathbb{R}$ whose characteristic functions $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfy

$$\varphi_n(t) \geq 1 - C|t|^{\beta} \text{ for all } n \in \mathbb{N}, t \in \mathbb{R},$$

for some constants $C > 0$ and $\beta > 1$. Show that $\{\mu_n\}_{n\in\mathbb{N}}$ is tight. (*Hint.* Prove and the use the following identity: $|x| = \frac{1}{K}\int_0^{\infty} \frac{1-\cos(tx)}{t^2}\,\mathrm{d}t$ for some $K > 0$.)

Problem 3. Let (X, Y) be a jointy Gaussian random vector, where $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^m$, with $m, n \in \mathbb{N}$. Assume, further, that $\mathbb{E}[X] = 0$ and $\mathbb{E}[Y] = 0$, and that the covariance matrix of Y is invertible. Prove that the conditional expectation $\mathbb{E}[X \mid \sigma(Y)]$ is given by a linear transformation of Y.