ALGEBRA PRELIMINARY EXAM: PART I

Each problem is worth 7 points and the passing score is 20 points.

Problem 1

Let G be a group of order 56. Prove that

a) G has a normal p-Sylow subgroup for some prime p dividing 56.
b) G is solvable.

Problem 2

Let n be odd. Consider the set of n-cycles in S_n. Prove that this set breaks up as the disjoint union of two conjugacy classes of equal size in A_n.

Problem 3

Let R be an integral domain such that every prime ideal of R is principal.

a) Consider the set of ideals of R which are not principal. Prove that if this set is non-empty, then it contains an element I which is maximal under inclusion.
b) Prove that R is a principal ideal domain. (Hint: Show that I is principal.)

Problem 4

Consider the 2-dimensional $\mathbb{R}$ vector space $V = \mathbb{R}^2$ and the linear transformation $T : V \to V$ which is the projection onto the y-axis. View V as an $\mathbb{R}[x]$-module where $x \cdot v = T(v)$.

a) Determine all $\mathbb{R}[x]$-submodules of V.
b) Describe the structure of V as an $\mathbb{R}[x]$-module.

Date: August 19, 2026.