ALGEBRA PRELIMINARY EXAM: PART II

Each problem is worth 10 points and the passing score is 20 points.

Problem 1

Consider $f(x) = x^5 - 15x + 5$.

a) Determine the degree of the splitting field of $f(x)$ over $\mathbb{F}_2$.
b) Let F be the splitting field of $f(x)$ over $\mathbb{Q}$. Prove that $\mathrm{Gal}(F/\mathbb{Q}) \simeq \mathbf{S}_5$.
c) Determine whether $f(x)$ is solvable by radicals over $\mathbb{Q}$.

Problem 2

Let p be an odd prime and $F = \mathbb{Q}(\sqrt[p]{2})$.

a) Prove that F is not a subfield of any cyclotomic extension of $\mathbb{Q}$.
b) Determine a primitive generator of the Galois closure of F over $\mathbb{Q}$.

Problem 3

Let p be a prime, $\mathbb{F}_p$ the finite field with p elements, and $K = \mathbb{F}_p(x, y)$ where x, y are independent transcendentals over $\mathbb{F}_p$. Consider the following two subfields of L: $E = \mathbb{F}_p(x^p, y^p)$ and $F = \mathbb{F}_p(x^p - x, y^p - x)$.

a) Prove that K/E is not simple.
b) Determine the separable and inseparable degrees of K/E and K/F.
c) Prove that K contains a subfield L which is purely inseparable over F.

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