DIFFERENTIAL TOPOLOGY EXAM

AUGUST 19, 2026

You should attempt all three problems. The exam length is two hours. If a question is unclear, please ask. Although you should strive for neat and detailed solutions, we are mindful that the exam is timed.

Problem 1. Let $O(n) = \{A \in \mathrm{Mat}_{n\times n}(\mathbb{R})\colon A^TA = I\}$ be the group of orthogonal matrices and $\mathrm{Sym}(n) \subset \mathrm{Mat}_{n\times n}(\mathbb{R})$ be the space of symmetric matrices.

(a) Prove that $O(n)$ is a submanifold of $\mathrm{Mat}_{n\times n}(\mathbb{R})$.
(b) Prove that at I, the intersection of $O(n)$ and $\mathrm{Sym}(n)$ is transverse.

Problem 2. Consider the map $f\colon \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = z + z^2.$$

(a) Show that f extends to a smooth map $\widetilde{f}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$.
(b) Is $\widetilde{f}$ Lefschetz?
(c) Calculate the Lefschetz number of $\widetilde{f}$. Use only tools from differential (rather than algebraic) topology. Make sure to take into account the point at ∞.

Problem 3. Fix a standard affine chart $U \cong \mathbb{R}^3$ on $\mathbb{RP}^3$ with affine coordinates (x, y, z). On this chart, consider the smooth 3-form

$$\omega = \frac{1}{(1 + x^2 + y^2 + z^2)^2}\, dx \wedge dy \wedge dz.$$

(a) Show that ω extends to a smooth 3-form $\widetilde{\omega}$ defined on all of $\mathbb{RP}^3$.
(b) Does there exist a smooth 2-form α defined on U such that $\omega = d\alpha$ on U?
(c) Does there exist a smooth 2-form $\widetilde{\alpha}$ defined on all of $\mathbb{RP}^3$ such that $\widetilde{\omega} = d\widetilde{\alpha}$ on $\mathbb{RP}^3$?