Preliminary Examination: Algebraic topology. August 18, 2026

Instructions: Answer all three questions. All questions (but not all subparts) carry equal weight

Time limit: 2 hours.

1. Let M and N be closed connected surfaces and let $X = M \vee N$ be obtained by identifying a point $m \in M$ with a point $n \in N$. Denote the identified point by x_0.

(a) Compute the local homology groups $H_k(X, X - \{x\})$ when $x \neq x_0$.

(b) Compute $H_k(X, X - \{x_0\})$.

(c) Prove that X is not a surface.

(d) Suppose M and N are not homeomorphic. Show that every homeomorphism $X \to X$ preserves each of M and N.

2. Given a map $f : X \to Y$, recall that the mapping cone C_f is obtained from $X \times I \sqcup Y$ by collapsing $X \times \{0\}$ to a point and by gluing $X \times \{1\}$ to Y via f, i.e. $(x, 1) \sim f(x)$. Consider the mapping cone C_f for the case that X and Y are both copies of the torus $S^1 \times S^1$ and the map $f : S^1 \times S^1 \to S^1 \times S^1$ is defined by $f(z, w) = (z^2, w^3)$.

(a) Compute the fundamental group $\pi_1(C_f)$.

(b) Compute all singular homology groups of C_f (with integer coefficients).

(c) Is the Y torus a retract of C_f?

3. Let $X = S^1 \vee S^1$, with the wedge point denoted by x_0. Then $\pi_1(X, x_0) = \langle a, b\rangle$ is the free group on two generators, namely a loop a traversing the first S^1 and a loop b traversing the second S^1.

Let $\widetilde{X}$ be the directed graph, embedded in the plane, whose vertices are the integer lattice points $\mathbb{Z} \times \mathbb{Z}$, and with a directed horizontal edge $(m, n) \to (m + 1, n)$ and a directed vertical edge $(m, n) \to (m, n + 1)$ for all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. Let $p : \widetilde{X} \to X$ be the covering space defined by mapping all of the vertices to x_0, the horizontal edges to the a circle, and the vertical edges to the b circle, preserving orientation. You do not need to check this is a covering space.

(a) Determine the group of deck transformations of p. Is p a regular covering space?

(b) Determine exactly which elements of $\langle a, b\rangle$ lift to loops at $(0, 0)$. Hence identify the subgroup $H = p_*\pi_1(\widetilde{X}, (0, 0)) < \pi_1(X, x_0)$. Explain why H is isomorphic to a free group and find the rank.

(c) Suppose $q : Y \to X$ is a regular cover with abelian deck transformation group. Show that p factors through q, i.e. that there exists a covering map $p' : \widetilde{X} \to Y$ such that $p = q \circ p'$.