PRELIMINARY EXAMINATION IN ANALYSIS

PART I

FALL 2026

Please provide complete proofs for at least 4 of the following 5 problems.

(1) Let $f \in L^1(\mathbb{R}, \lambda)$ and recall that the maximal function is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)| \, dy.$$

Prove the maximal inequality: there exists a constant $C > 0$ such that

$$\lambda(\{x \in \mathbb{R} : \ Mf(x) > t\}) \leq C \frac{\|f\|_1}{t}.$$

Hint: you can use the covering theorem which says that if $\mathcal{B}$ is any collection of balls in $\mathbb{R}$ with uniformly bounded radii then there exists a pairwise-disjoint sub-collection $\mathcal{F} \subset \mathcal{B}$ such that $\cup\{B : \ B \in \mathcal{B}\} \subset \cup\{4B : \ B \in \mathcal{F}\}$ where $4B$ is the ball with the same center as B and 4 times the radius.

(2) Let ν be a finite Borel measure on $[0, 1]$. Define

$$f(x) = \nu([0, x)).$$

Prove ν is absolutely continuous to Lebesgue measure if and only if f is absolutely continuous.

(3) Construct a sequence $\mu_1, \mu_2, \ldots$ of Borel probability measures on $\mathbb{R}$ which are all absolutely continuous to Lebesgue measure and which converge to the Dirac mass δ_0 in the weak* topology on $C_0(\mathbb{R})^*$. Prove your answer.

(4) Let f be a continuous function on $[0, 1]$. What is

$$\lim_{n\to\infty} n \int_0^1 x^n f(x) \, dx?$$

Hint: Let $0 < \delta < 1$ and write

$$n \int_0^1 x^n f(x) \, dx = n \int_0^{1-\delta} x^n f(x) \, dx + n \int_{1-\delta}^1 x^n f(x) \, dx.$$

(5) Prove that $\ell^1(\mathbb{N})$ is not reflexive. In other words, the canonical injection $\ell^1(\mathbb{N}) \to \ell^\infty(\mathbb{N})^*$ is not surjective.