PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part I

Wednesday, August 19, 2026, 12:00 pm–2:00 pm

Work all 3 of the following 3 problems.

1. Let $(X, ||\cdot||)$ be a Banach space and let $Y \subset X$ $(Y \neq X)$ be a subspace. We will say that $y \in Y$ is a *best approximation* of a given $x \in X$ if it satisfies

$$||y - x|| = \inf_{z \in Y} ||z - x||.$$

(**a**) If X is reflexive and Y is finite-dimensional, show that a best approximation $y \in Y$ exists for each $x \in X$.

(**b**) For a given $x \in X$, show that the set of all best approximations is convex.

(**c**) Show that the best approximation need not be unique by considering $X = (\mathbb{R}^2, \|\cdot\|_{\ell^\infty})$ and Y being the subspace $\{(t, 0) : t \in \mathbb{R}\}$.

2. Let the normed linear space X be *uniformly convex*, which means that given any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ satisfying $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, it holds that

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

(**a**) Show that if $x, y \in X$, $x \neq y$, and $\|x\| = \|y\| = 1$, then $\|x + y\| < 2$.

(**b**) Show that X is *strictly convex:* $\|x + y\| < \|x\| + \|y\|$ whenever x and y are linearly independent. You may use the inequality

$$\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \geq \left\| \frac{x}{\|x\|} + \frac{y}{\|x\|} \right\| - \left\| \frac{y}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

(**c**) If H is a Hilbert space, show that H is uniformly convex. [Hint: use the parallelogram law.]

3. Suppose that X is a Banach space and $T \in C(X, X)$ is compact.

(**a**) Define what it means for T to be compact.

(**b**) Prove that $T \in B(X, X)$.

(**c**) Prove that if $S \in B(X, X)$, then TS and ST are compact.

(**d**) If $S = (I - T)^{-1}$, show that $I - S$ is compact.