PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part II

Friday, August 21, 2026, 9:00 am–11:00 am

Work all 3 of the following 3 problems.

1. Consider the partial differential equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty,\ 0 < y < \infty, \\ u(x,0) = f(x) \text{ and } u(x,y) \to 0 \text{ as } x^2 + y^2 \to \infty, \end{cases}$$

for the unknown function $u(x,y)$, where f is a nice function (say f is in the Schwartz class).

(**a**) The Fourier transform of $f \in L^1(\mathbb{R})$ is $\mathcal{F}f(\xi) = \hat{f}(\xi) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx$. When $f(x) = e^{-a|x|}$, $a > 0$, one has $\mathcal{F}f(\xi) = \sqrt{\dfrac{2}{\pi}} \dfrac{a}{a^2 + \xi^2}$. Find the Fourier transform of $\dfrac{1}{1+x^2}$.

(**b**) Find a function $g(x,y)$ such that $u(x,y) = f * g(x,y) = \displaystyle\int_{\mathbb{R}} f(z)\, g(x-z,y)\, dz$.

2. Let H and W be real Hilbert spaces, and $V \subset H$ a closed linear subspace. Let $A : H \to H$ and $B : V \to W$ be bounded linear operators, where V has the norm $\|v\|_V = \|v\|_H + \|Bv\|_W$. For any $f \in V$ and $0 \le \delta < 1$, consider the problem: Find $(u,p) \in V \times W$ such that

$$\mathcal{B}((u,p),(v,w)) = \langle f, v\rangle_H \quad \forall (v,w) \in V \times W,$$

where the bilinear form is

$$\mathcal{B}((u,p),(v,w)) = \langle Au, v\rangle_H - \langle B^* p, v\rangle_H + \langle Bu, w\rangle_W + \langle p, w\rangle_W + \delta \langle Bu + p, Bv\rangle_W.$$

Assume that A is coercive on H but not on V (i.e., $\alpha \|v\|_H^2 \le \langle Av, v\rangle_H$ for some $\alpha > 0$).

(**a**) Show that $\mathcal{B}$ is continuous on $V \times W$. Use the proper norms!

(**b**) Show that $\mathcal{B}$ is coercive on $V \times W$ when $0 < \delta < 1$.

(**c**) Show that there is a solution to the problem for $\delta > 0$, and then for $\delta = 0$. [Hint: for the latter, replace w by $w - \delta Bv$.]

3. Let X and Y be normed linear spaces.

(**a**) Let $X = \mathbb{R}$ and $f : \mathbb{R} \to Y$ be a function which is continuous on the segment $[a,b]$ and differentiable on the segment (a,b), and let $A \in B(\mathbb{R}, Y)$ be given. Show that

$$||f(b) - f(a) - A(b-a)||_Y \le M||b-a||_{\mathbb{R}}, \quad \text{where} \quad M = \sup_{x \in (a,b)} ||Df(x) - A||_{B(\mathbb{R},Y)}.$$

(**b**) Let $X = \mathbb{R}$ and $g : \mathbb{R} \to Y$ be a function which is continuous in $\mathbb{R}$ and differentiable in $\mathbb{R} \setminus \{a\}$. Show that, if $L = \lim_{x \to a} Dg(x)$ exists, then g is differentiable at a and $Dg(a) = L$.

(**c**) Consider a general NLS X and $g : X \to \mathbb{R}$ where $g(x) = ||x||_X$. Show that g cannot be differentiable at $x = 0$.