The University of Texas at Austin
Probability Theory I - Prelim Exam

Aug 18, 2026, 12:00 noon - 2:00 pm

Problem 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a random variable U with the uniform distribution on $[0,1]$ is defined. Show that the a.s. convergence is not metrizable, i.e., there exists no metric on the space $\mathbb{L}^0$ of a.s.-equivalence classes of random variables such that

$$d(X_n, X) \to 0 \text{ is equivalent to } X_n \xrightarrow{\text{a.s.}} X.$$

for every sequence $\{X_n\}_{n\in\mathbb{N}}$ and every X in $\mathbb{L}^0$. Is the same necessarily true if the existence of U is not required?

(*Hint:* Show, and use the fact that in a metric space the following are equivalent for a sequence $\{x_n\}_{n\in\mathbb{N}}$ 1) $x_n \to x$, and 2) every subsequence of $\{x_n\}_{n\in\mathbb{N}}$ admits a further subsequence which converges to x.)

Problem 2. Let $X_1, X_2, \dots$ be iid standard Cauchy random variables, i.e., with density $f(x) = \frac{1}{\pi(1+x^2)}$ and characteristic function $\varphi(t) = e^{-|t|}$, and let $\overline{X}_n = \frac{X_1 + \dots + X_n}{n}$.

1. Show that $\overline{X}_n$ has the standard Cauchy distribution, for every $n \in \mathbb{N}$.
2. Show that $\overline{X}_n$ does not converge in probability to any constant as $n \to \infty$.
3. Show that $\overline{X}_n$ does not converge a.s. to any random variable as $n \to \infty$.

Problem 3. Let Θ be uniformly distributed on $[0,1]$ and, conditionally on $\Theta = \theta$, let $X_1, X_2, \dots$ be iid Bernoulli random variables with parameter θ; that is,

$$\mathbb{P}[X_1 = x_1, \dots, X_n = x_n \mid \sigma(\Theta)] = \Theta^s (1-\Theta)^{n-s}, \quad s = x_1 + \dots + x_n,$$

for every $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in \{0,1\}$.

1. Find the conditional distribution and the conditional expectation of Θ given $\sigma(X_1, \dots, X_n)$.
2. Compute $\mathbb{P}[X_{n+1} = 1 \mid \sigma(X_1, \dots, X_n)]$ and show that the sequence $X_1, X_2, \dots$ is exchangeable but not independent.

(*Note: Exchangeable* means that $(X_1, \dots, X_n) \overset{(\mathrm{d})}{=} (X_{\pi_1}, \dots, X_{\pi_n})$ for all n and all permutations π of $\{1, \dots, n\}$. The identity $\int_0^1 \theta^a (1-\theta)^b \, \mathrm{d}\theta = \frac{a!\, b!}{(a+b+1)!}$, valid for $a, b \in \mathbb{N}_0$, may be used without proof.)