ALGEBRA PRELIMINARY EXAM: PART I

Problem 1

a) Let G be a finite group. Assume the intersection of all the non-trivial subgroups of G is non-trivial. Suppose G acts faithfully on the finite set X (recall this means the map $G \to \mathrm{Perm}(X)$ is injective). Show $|X| \geq |G|$.
b) Let Q be the quaternion group (of order 8). Find the smallest n such that Q is isomorphic to a subgroup of the symmetric group S_n.

Problem 2

Let F be the set of Sylow 5-subgroups of the symmetric group S_5.

a) Prove the elements of F are exactly the subgroups generated by 5 cycles, and $|F| = 6$. Explain why this gives an injective homomorphism $\varphi : S_5 \to S_6 = \mathrm{Perm}(F)$, with image a transitive subgroup (i.e., one that acts transitively on the 6 elements of F). Set $H := \mathrm{im}(\varphi) \subset S_6$. Prove $|S_6/H| = 6$.
b) Prove that H is not conjugate to any of the obvious $S_5 \subset S_6$ (the stabilizers of one of the elements of F).
c) Prove that S_6 has an outer automorphism, ie an automorphism which is not given by conjugation.

You may use without proof the standard fact that $A_n \subset S_n$ is the only non-trivial proper normal subgroup for any $n \neq 4$.

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Problem 3

Let S be a commutative ring. We say that a non-invertible $s \in S$ is *indecomposible* if $s = a \cdot b$ for $a, b \in S$ implies at least one of a, b is invertible.

a) Let S be an integral domain, and $s \in S$ a non-invertible element. Show s is indecomposible iff (s) is maximal among proper principal ideals (i.e., ideals generated by a single non-invertible element).

b) Show that in a Noetherian integral domain, any element is a product of indecomposible elements.

Problem 4

Let S be a commutative integral domain. Suppose every finitely generated torsion free S module is free. Prove that S is a PID.