ALGEBRA PRELIMINARY EXAM: PART II

Problem 1

Let $\alpha = \sqrt{3} + \sqrt{5}$.

a) Determine the minimal polynomial $p(x)$ of α over $\mathbb{Q}$
b) Prove that $\sqrt[3]{5} \notin \mathbb{Q}(\alpha)$.
c) Prove that $\mathbb{Q}(\alpha)$ is Galois over $\mathbb{Q}$.

Problem 2

Let p be a prime, and $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$

a) Determine the number of irreducible polynomials of degree 5 in $\mathbb{F}_p[x]$.
b) Prove that $x^4 - 10x^2 + 1$ is irreducible in $\mathbb{Z}[x]$ but reducible in $\mathbb{F}_p[x]$ for every prime p.

State clearly all results about the structure of finite fields that you use in solving these problems.

Problem 3

a) Prove that if the Galois group of the splitting field of a cubic polynomial over $\mathbb{Q}$ is of degree 3 then all the roots of the cubic polynomial are real.
b) Determine the Galois group of $x^3 - x + 1$ as a subgroup of S_3.
c) Let F be the splitting field of $x^3 - x + 1$ over $\mathbb{Q}$, and ζ_n be a primitive n-th root of unity. Does there exist $n \in \mathbb{N}$ such that $F \subseteq \mathbb{Q}(\zeta_n)$?

You may use without proof the fact that the discriminant of $x^3 + ax + b$ is $-4a^3 - 27b^2$.

Date: January 12, 2024.