Differential Topology Prelim Exam
January 10, 2024, 11:30–1:30

Solve all three problems.

Problem 1. True or false with brief explanations/examples/counterexamples.

(a) Suppose Y is a linear subspace of $\mathbb{R}^n$, X is a smooth manifold, $f : X \to \mathbb{R}^n - \{0\}$ is smooth, and $x \in X$. Then f is transverse to Y at x, if and only if $\bar{f}$ is transverse to $PY \subseteq \mathbb{R}P^{n-1}$ at x. Here $\bar{f}$ is f followed by the usual map $\mathbb{R}^n - \{0\} \to \mathbb{R}P^{n-1}$.

(b) Suppose $f : X \to Y$ is a smooth map of manifolds, and $y \in Y$. Then f is a regular value of f if and only if f is a local diffeomorphism near each $x \in f^{-1}(y)$.

(c) Every injective immersion is a diffeomorphism onto a submanifold.

Problem 2. Let ω be the 2-form on $\mathbb{R}^3$ given in standard coordinates by

$$\omega = dx \wedge dz + z\, dx \wedge dy$$

(a) Prove that ω is neither closed nor translation-invariant.

(b) Nevertheless, if S is an oriented surface embedded in $\mathbb{R}^3$, then $\int_S \omega = \int_{S_v} \omega$ for any vector $v \in \mathbb{R}^3$, where S_v and its orientation are got by translating S by v.

Problem 3. Let $\lambda_0, \ldots, \lambda_3 \in \mathbb{R} - \{0\}$, and let f be the following self-map of $\mathbb{R}P^3$, given in homogeneous coordinates by

$$f\big([x_0 : \cdots : x_3]\big) = [\lambda_0 x_0 : \cdots : \lambda_3 x_3]$$

(a) Under what conditions is f a Lefschetz map?

(b) Under these conditions, work out the fixed points of f, and its local degrees and Lefschetz numbers there.

(c) What are the degree and Lefschetz number of f?