PRELIMINARY EXAMINATION IN ANALYSIS
PART I
AUGUST 2024

Please provide complete proofs for at least 3 of the following 4 problems.

(1) Let $(X, \mathcal{B}, \mu)$ be a measure space with $\mu(X) < \infty$. For $n \in \mathbb{N}$, let $f_n : X \to \mathbb{R}$ be a measurable function. Suppose f_n converges to a function f pointwise a.e. as $n \to \infty$. Prove that f_n converges to f in measure.

(2) Let f and g be real-valued integrable functions on a measure space (X, μ) and let

$$F_t = \{x \in X : \ f(x) > t\}$$
$$G_t = \{x \in X : \ g(x) > t\}.$$

Prove that

$$\|f - g\|_1 = \int_{-\infty}^{\infty} \mu(F_t \triangle G_t)\,\mathrm{dt}$$

where

$$F_t \triangle G_t = (F_t \setminus G_t) \cup (G_t \setminus F_t)$$

is the symmetric difference.

(3) Let H be an infinite-dimensional separable real Hilbert space. Let

$$S = \{x \in H : \ \|x\| = 1\}$$
$$B = \{x \in H : \ \|x\| \leq 1\}$$

be the unit-sphere and ball respectively. Prove that S is weakly dense in B. This means that for any $y \in B$ there exists a sequence $(x_n)_n \subset S$ such that for any $z \in H$, $\lim_{n\to\infty}\langle x_n, z\rangle = \langle y, z\rangle$.

(4) Let $f : [0,1] \to [0,1]$ be the Cantor function. That is, if $C = \{x = \sum_{k=1}^{\infty} x_k 3^{-k} : \ x_k \in \{0,2\}\}$ is the middle thirds Cantor set then

$$f\left(\sum_{k=1}^{\infty} x_k 3^{-k}\right) = \sum_{k=1}^{\infty} (x_k/2) 2^{-k}$$

on C, f is constant on each interval in the complement of C and f is continuous.

(a) Is f uniformly continuous?
(b) Is f of bounded variation?
(c) Is f absolutely continuous?
(d) Let λ_f be the Lebesgue-Stieltjes measure of f. Is λ_f absolutely continuous to Lebesgue measure, singular to Lebesgue measure or neither?