Preliminary Examination: Algebraic topology. August 20, 2024

Instructions: Answer all three questions. All questions (but not all subparts) carry equal weight

Time limit: 2 hours.

1. This problem is about constructing maps $S^n \to S^n$ of every possible degree.

(a) For each $k \in \mathbb{Z}$ construct a continuous map $f_k : S^1 \to S^1$ of degree k. Prove carefully that $\deg f_k = k$.

(b) Let $n \geq 1$, and let $f : S^n \to S^n$ be a continuous map. Explain how to use the suspension operation to construct a map $Sf : S^{n+1} \to S^{n+1}$ and prove that $\deg Sf = \deg f$. [*Hint*: It may helpful to consider the cone CS^n.]

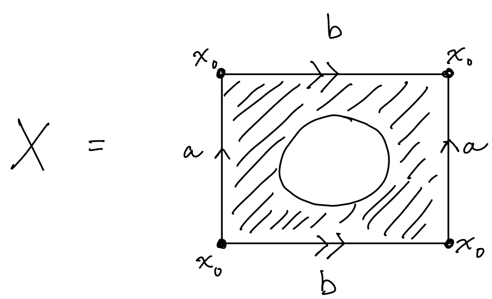
(c) Prove that for all $n \geq 1$ and $k \in \mathbb{Z}$ there exists a map $S^n \to S^n$ of degree k.

2. Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle. Fix a pair (p, q) of relatively prime integers. Consider the quotient space $X = S^1 \times S^1 \times [0, 1]/\sim$, where the equivalence relation $\sim$ is given by $(x, y, 0) \sim (x', y', 0)$ whenever $x = x'$, and $(x, y, 1) \sim (x', y', 1)$ whenever $px + qy = px' + qy'$. For the following questions, your answer may depend on p, q.

(a) Compute $\pi_1(X)$.

(b) Compute all singular homology groups of X (with integer coefficients).

3. Let X be the orientable surface of genus one with one boundary component, i.e. X is a torus with a disk removed. The fundamental group of X is isomorphic to the free group of rank two, generated by elements a, b as indicated in the diagram: $\pi_1(X, x_0) = \langle a, b \rangle$.



Now consider the dihedral group of order eight, $D_8 = \langle x, y : x^2 = y^4 = 1, xyx^{-1} = y^{-1} \rangle$, and let $h : \pi_1(X, x_0) \to D_8$ be the homomorphism given by $h(a) = x,\ h(b) = y$. Let $H = \ker h < \pi_1(X, x_0)$. Let $p_H : \widetilde{X}_H \to X$ be the associated covering space.

(a) What is the degree of the cover p_H? What is the Euler characteristic of $\widetilde{X}_H$?

(b) The space $\widetilde{X}_H$ is an orientable surface of some genus g and some number of boundary components k. What are k and g?