ALGEBRA PRELIMINARY EXAM: PART I

The first three problems are worth 10 points, and the last one 5 points. The passing score is 23 points.

(1) Prove or disprove: there is a simple group of order 2025.

(2) Let k be a field and let $A = k[X,Y]/(X^2, XY, Y^2)$.
   (a) Explicitly determine the invertible elements of A.
   (b) Explicitly classify the principal[1] ideals of A.

(3) (a) Find the invariant factors of the quotient group $\mathbb{Z}^3/N$, where N is the subgroup of $\mathbb{Z}^3$ generated by $(-4,4,2)$, $(16,-4,-8)$, and $(8,4,2)$.
   (b) Let $\mathbb{F}_2$ be the field of 2 elements and let $R = \mathbb{F}_2[X]$. List, up to isomorphism, all R-modules with exactly 16 elements.

(4) Does there exist a 3×3 matrix A over $\mathbb{Q}$ such that $A^8 = I$ but $A^4 \neq I$?

[1]Just saying ideals of the form (a) for $a \in A$ is not enough: also determine when does $(a) = (b)$ for $a, b \in A$.