Differential Topology Prelim Exam

January 11, 2025

Instructions: There are three problems, each worth 10 points. Do all three. You have two hours.

Problem 1 Let M be a smooth m-dimensional manifold embedded in $\mathbb{R}^n$. Let p be a point in the complement of M. Prove that there exists a line L through p such that $L \pitchfork M$. (That is, the intersection of L and M is transverse.) The dimensions m and n are intended to be arbitrary, except that $n > 0$ and $m \leq n$.

Problem 2 On $\mathbb{R}^4$, consider the linear map $f_0 : \mathbb{R}^4 \to \mathbb{R}^4$ given by $f_0(w, x, y, z) = (-w, x, 2y, 3z)$. This induces a map $f : \mathbb{R}P^3 \to \mathbb{R}P^3$ by $f[w, x, y, z] = [f_0(w, x, y, z)]$, where $[w, x, y, z]$ is the equivalence class of (w, x, y, z).

a. Find all of the fixed points of f.

b. For each fixed point, compute the local Lefschetz number.

c. Either prove that f is homotopic to the identity map or prove that it isn't.

Problem 3 On $\mathbb{R}^2 - \{0\}$, consider the 1-form

$$\omega = \frac{x\,dy - y\,dx}{2\pi(x^2 + y^2)}.$$

a. Show that ω is closed.

b. Is ω exact? Either find a function $f : \mathbb{R}^2 - \{0\} \to \mathbb{R}$ such that $\omega = df$ or prove that no such function exists.

c. Let $S^1 = [0, 1]/\sim$, where the points 0 and 1 are identified, and let $\gamma : S^1 \to \mathbb{R}^2 - \{0\}$ be a closed curve that is homotopic to the curve $\gamma_n(t) = (\cos(2\pi nt), \sin(2\pi nt))$ that wraps around the unit circle n times. Show that $\int_{S^1} \gamma^*\omega = n$