ALGEBRA PRELIMINARY EXAM: PART II

Each problem is worth 10 points and the passing score is 20 points.

Problem 1

Let p be a prime, $\mathbb{F}_p$ be the field with p elements.

a) Let $f(x) = x^p - x + 1 \in \mathbb{F}_p(x)$ and α be a root of $f(x)$. Prove that $\mathbb{F}_p(\alpha)/\mathbb{F}_p$ is Galois and determine the cardinality of $\mathbb{F}_p(\alpha)$.
b) Prove that $\mathbb{F}_p(x, y)/\mathbb{F}_p(x^p, y^p)$ is not a simple extension.

Problem 2

Consider $f(x) = x^4 - 2x^2 - 2 \in \mathbb{Q}[x]$. Let $\pm\alpha, \pm\beta$ denote the roots of $f(x)$.

a) Determine the degree of the splitting field E of $f(x)$ over $\mathbb{Q}$.
b) Prove that $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to the dihedral group of order 8.
d) Determine a primitive generator of $E/\mathbb{Q}$.
c) Determine all the subfields of E and identify the ones that are Galois over $\mathbb{Q}$.

Problem 3

Let K be a field and $f(x) \in K[x]$ be a separable irreducible polynomial of degree 5 with distinct roots α and β. Prove that if $K(\alpha) = K(\beta)$ then $K(\alpha)/K$ is Galois.

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