PRELIMINARY EXAMINATION IN ANALYSIS
PART I
JANUARY 2025

Work all 4 of the following 4 problems.

(1) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$ and let f be a real valued non-negative measurable function on X. Show there exists a non-negative function g on $(0, \infty)$ such that g is monotone decreasing and

$$\int_X |f|^p \, d\mu = \int_0^\infty g(t)^p \, dt$$

for every $p \geq 1$.

(2) Let H be an infinite-dimensional separable Hilbert space. Does there exist a subset $C \subset H$ such that (a) C is closed in the norm topology and (b) there does not exist an element $x \in C$ with $\|x\| = \inf\{\|y\| : \ y \in C\}$? Prove your answer.

(3) Let f be non-decreasing function on $[0, 1]$. You may assume that f is differentiable almost everywhere. Prove that

$$\int_0^1 f'(t) \, dt \leq f(1) - f(0).$$

Do not simply say that this comes from a theorem in a book.

(4) For $n \in \mathbb{N}$ and $0 \leq j \leq 2^n - 1$, let

$$I_{n,j} = [j2^{-n}, (j+1)2^{-n}).$$

For $f \in \mathrm{L}^1([0, 1))$ let $E_n f \in \mathrm{L}^1([0, 1))$ be the function

$$E_n(f)(x) = 2^n \int_{I_{n,j}} f \, dt$$

where j is the unique number so that $x \in I_{n,j}$. Prove that for Lebesgue almost every $x \in [0, 1)$,

$$\lim_{n \to \infty} E_n(f)(x) = f(x).$$

Do not use the Martingale Convergence Theorem. Instead use material usually covered in a Real Analysis class.