PRELIMINARY EXAMINATION: ANALYSIS — Part II

January 9, 2025

Work all 4 of the following 4 problems.

1. For which $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{C})$ are the Möbius transformations

$$T_A : z \mapsto \frac{az+b}{cz+d}$$

automorphisms of $\mathbb{C}$, respectively automorphisms of $\mathbb{C}_\infty$?

2. Assume that $f : \Omega \to f(\Omega)$ is holomorphic and injective on $\Omega \subset \mathbb{C}$.
(a) Prove that $f' \neq 0$ everywhere in Ω.
(b) Prove that f is an open map.
(c) Prove that $f^{-1} : f(\Omega) \to \Omega$ is holomorphic.

3. By integrating $(e^{2iz} - 1)/z^2$ around a suitable contour, evaluate

$$\int_0^\infty \frac{\sin^2(x)}{x^2}\, dx.$$

4. Prove that

$$\frac{\sin(z)}{\sin(\pi z)} = \frac{1}{\pi} + \frac{z}{\pi} \sum_{n \neq 0} (-1)^n \frac{\sin(n)}{n(z-n)}, \qquad z \in \mathbb{C} \setminus \mathbb{Z}.$$