PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part I

Friday, January 10, 2025, 11:30 am-1:30 pm

Work all 4 of the following 4 problems.

1. [15 points] Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space and let $S \subset H$ be a closed convex subset. For fixed $x \in H$ and $y \in S$, show that

$$\|x - y\| = \inf_{z \in S} \|x - z\| \qquad \text{if and only if} \qquad \langle x - y, z - y \rangle \leq 0 \text{ for all } z \in S.$$

[Hint: for the direct implication, consider $\alpha \in (0, 1)$ and $y + \alpha(z - y)$.]

2. [15 points] Suppose X is a normed linear space and Y a Banach space. Let $\{T_n\}_{n=1}^{\infty}$ be a collection of compact linear operators taking X to Y that is also convergent in norm to $T \in B(X, Y)$, i.e.,

$$\|T_n - T\| \longrightarrow 0 \quad \text{as } n \to \infty.$$

Show that T is also a compact operator. [Hint: use a diagonalization argument. Start from a bounded sequence $\{x_n\}_{n=1}^{\infty}$ from which you can extract a convergent subsequence from the images of the points.]

3. [10 points] Show that $\mathrm{PV}(1/x)$ is a distribution. Recall that for $\phi \in \mathcal{D}(\mathbb{R})$,

$$\langle \mathrm{PV}(1/x), \phi \rangle = \lim_{\epsilon \to 0^+} \int_{|x| > \epsilon} \frac{1}{x}\, \phi(x)\, dx.$$

4. [20 points] Let (X, d) be a metric space. For $n = 1, 2, \ldots$, let $A_n \subset X$ be a nonempty, closed subset such that $A_{n+1} \subset A_n$ and $\mathrm{diam}(A_n) \to 0$ as $n \to \infty$.

(**a**) If X is complete, show that there is a unique $x \in X$ such that $\displaystyle\bigcap_{n=1}^{\infty} A_n = \{x\}$.

(**b**) Conversely, show that X is complete if every sequence $\{A_n\}_{n=1}^{\infty}$ with the above properties also satisfies $\displaystyle\bigcap_{n=1}^{\infty} A_n = \{x\}$.