PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part II

Wednesday, January 8, 2025, 11:30am-1:30pm

Work all 3 of the following 3 problems.

1. (20 pts) For $f \in S(\mathbb{R})$, define the Hilbert transform of f by $Hf = PV\left(\frac{1}{\pi x}\right) * f$, where the convolution uses ordinary Lebesgue measure.

1) Show that $PV\left(\frac{1}{x}\right) \in \mathcal{S}'$.

2) Using the fact the Fourier Transform $F\left(PV\left(\frac{1}{x}\right)\right) = -i\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}(\xi)$, where $\mathrm{sgn}(\xi)$ is the sign of ξ, show that

$$\|Hf\|_{L^2} = \|f\|_{L^2} \quad \text{and } HHf = -f, \text{ for } \ f \in S(\mathbb{R}).$$

3) Extend H to $L^2(\mathbb{R})$.

2. (20 pts) Let $f \in L^2\left(\mathbb{R}^d\right)$ and consider the problem

$$-\Delta u + u = f \qquad \text{in } \mathbb{R}^d\,.$$

i. Find the variational problem associated to the PDE.

ii. Use the Lax Milgram Theorem to show the existence and uniqueness of a solution in $H^1\left(\mathbb{R}^d\right)$ to the variational problem.

iii. Using the Fourier transform, show that the solution is actually in $H^2\left(\mathbb{R}^d\right)$.

3. (20 pts) For fixed $T > 0$, let $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous and Lipschitz continuous in the second argument, i.e., there is some $L > 0$ such that

$$\|g(t, v) - g(t, w)\| \leq L\|v - w\| \quad \forall v, w \in \mathbb{R}^d, t \in [0, T]$$

where $\|\cdot\|$ is the norm on $\mathbb{R}^d$. For any $u_0 \in \mathbb{R}^d$, consider the initial value problem (IVP) $u'(t) = g(t, u(t))$ and $u(0) = u_0$.

a) Write this IVP as the fixed point of a functional $G : C^0\left([0, T]; \mathbb{R}^d\right) \to C^0\left([0, T]; \mathbb{R}^d\right)$.

b) Normally, we use the $L^\infty([0, T])$-norm for $C^0\left([0, T]; \mathbb{R}^d\right)$. Show that the function $\|\!|\cdot|\!\| : C^0\left([0, T]; \mathbb{R}^d\right) \to [0, \infty)$, defined by

$$|\!|\!| v |\!|\!| = \sup_{0 \leq t \leq T} \left(e^{-Lt}\|v(t)\|\right)$$

is a norm equivalent to the $L^\infty([0, T])$-norm.

c) In terms of this new norm, show that G is a contraction.

d) Explain how we conclude that there is a unique solution $u \in C^1\left([0, \infty); \mathbb{R}^d\right)$ to the IVP for all time.