Prelim, Numerical Analysis I (fall course), 9:00-11:00 AM, 1/8/25

1. (a) Define Krylov subspace based on powers of a matrix A and a vector b. Give conditions for the maximal dimension of the Krylov subspace. (b) Define what is meant with a Krylov subspace method and determine which of Jacobi, Gauss-Seidel and the conjugate gradient methods are Krylov subspace methods. Motivate your answer. (c) The power method for finding maximal eigenvalues also generates powers of a matrix A times a vector b. Prove convergence of the power method under suitable conditions.

2. Consider the system of nonlinear equations, $f(x) = 0$, and an iterative method, $x_{n+1} = x_n + Af(x_n)$, for finding the solution. (a) Use fixed point iteration analysis for determining suitable conditions on A and f for convergence. (b) If f is linear, $f(x) = Bx - c$, what is an optimal A? Is this choice practical. (c) If A is allowed to be variable, $A = A_n$, show that quadratic convergence is possible.

3. (a) Derive the open Newton-Cotes formula for approximating, $\int_a^b f(x)dx$, based on two function evaluations and compare the degree of exactness with the open Newton-Cotes formula with one function evaluation, the midpoint rule. (b) Compare the result in (a) above with the degree of exactness for two-point Gaussian quadrature. (c) Define the Monte Carlo method for the integral in (a) and give an error estimate.